Final (+Solutions) (Due 3/19/21)

Your Name:

Names of Any Collaborators:

This final exam is graded out of 31 points and is worth 25% of your overall score in the class. This take-home exam is due by 11:59PM on 3/19/21, submitted via http://gradescope.com. Extensions will not be granted under any circumstances.

I expect your solutions to be *well-written, neat, and organized.* Do not turn in rough drafts. What you turn in should be the "polished" version of potentially several drafts.

The simple rules for the exam are:

1. You may freely use any theorems or problems that we have discussed in class, but you should make it clear where you are using a previous result and which result you are using. You may freely consult the textbook, the problem set, midterm, or quiz solutions, or my handwritten lecture notes.
2. You cannot use any results that we have not yet covered.
3. You are **NOT** allowed to consult external sources when working on the exam. This includes people outside of the class, other textbooks, and online resources.
4. You are **NOT** allowed to copy someone else's work.
5. You are **NOT** allowed to let someone else copy your work.
6. You are allowed to discuss the problems with each other and critique each other's work. Please name those with whom you have discussed the problems with.

If you break the rules, you will receive a zero on the exam. I will pursue anyone suspected of breaking these rules.

You should **turn in this cover page** together with all of the work that you have decided to submit.

To convince me that you have read and understand the instructions, sign (electronically) in the box below.

Signature:

Good luck and have fun!

F1. (**7 points**) For each statement, determine whether it is true or false. If it is true, briefly describe your reasoning (e.g. cite a theorem or sketch a proof). If it is false, provide a counterexample. Each statement is worth 1 point, awarded for correctly identifying the truth value (.5 points) and providing correct reasoning or counterexample (.5 points). No points will be awarded if you fail to correctly identify the truth value of a statement, or fail to follow these directions.

(i) Let $p(z)$ and $q(z)$ be nonzero polynomials with complex coefficient and no factors in common. Assume that $q(z)$ has $n \geq 1$ distinct zeros $q_1, \ldots, q_n$, each of which necessarily has order 1. It can be shown that $\frac{p(z)}{q(z)}$ has the partial fraction decomposition

$$\frac{p(z)}{q(z)} = g(z) + \frac{a_1}{z - q_1} + \frac{a_2}{z - q_2} + \cdots + \frac{a_n}{z - q_n} \tag{$*$}$$

where $g(z)$ is a polynomial with complex coefficients, and $a_1, \ldots, a_n \in \mathbb{C}$. The residue of $\frac{p(z)}{q(z)}$ at the isolated singularity $z = q_i$ is given by $\operatorname{Res}_{z=q_i} \frac{p(z)}{q(z)} = a_i$.[1]

Solution. This is **TRUE**. Fix $i \in \{1, \ldots, n\}$. To compute the residue of $\frac{p(z)}{q(z)}$ at $z = q_i$, we seek a Laurent series on the annulus $0 < |z - q_i| < R$ with $R = \min\{|q_i - q_j| : j \neq i\}$. Each of the functions

$$g(z), \frac{a_1}{z - q_i}, \ldots, \frac{a_{i-1}}{z - q_{i-1}}, \frac{a_{i+1}}{z - q_{i+1}}, \ldots, \frac{a_n}{z - q_n}$$

are analytic on the open disk $|z - q_i| < R$. Therefore, by Taylor's theorem, each of these functions has a Taylor series expansion about $z = q_i$. The Laurent series for $\frac{p(z)}{q(z)}$ is obtained by replacing each of these functions with it's Taylor series expansion in ($*$). Therefore, the principle part of $\frac{p(z)}{q(z)}$ is simply $\frac{a_i}{z - q_i}$ and of course it follows that $\operatorname{Res}_{z=q_i} \frac{p(z)}{q(z)} = a_i$. □

(ii) If $f(z)$ is analytic at $z_0 = 0$, then $f\left(\frac{1}{z}\right)$ has an essential singularity at $z_0 = 0$.

Solution. This is **FALSE**. Let $f(z) = z$. Then $f(z)$ is entire and hence analytic at $z_0 = 0$. Yet, $f(1/z) = \frac{1}{z}$ has a simple pole at $z_0 = 0$ (this is clear since the function is already in the form of a Laurent series). □

(iii) If $f(z)$ is entire and $f\left(\frac{1}{z}\right)$ has a pole at $z_0 = 0$, then $f(z)$ is a polynomial.

Solution. This is **TRUE**. If $f(z)$ is entire, then $f(z)$ has a MacLaurin series expansion that converges for all $z \in \mathbb{C}$:

$$f(z) = \sum_{n=0}^{\infty} a_n z^n. \tag{$*$}$$

But $f(1/z)$ has an isolated singularity at $z_0 = 0$ and the Laurent series on some annulus $0 < |z| < R$ is easily obtained from ($*$):

$$f(1/z) = \sum_{n=0}^{\infty} \frac{a_n}{z^n}.$$

[1] The statement to be proved or disproved here is the statement about the residues. You can assume that the partial fraction decomposition is correct.

But if $f(1/z)$ has a pole of order $m \geq 1$, then $a_n = 0$ for all $n > m$ and $a_m \neq 0$. Therefore,

$$f(z) = \sum_{n=0}^{m} a_n z^n = a_0 + a_1 z + a_2 z^2 + \cdots + a_m z^m.$$

□

(iv) If z_0 is an isolated singularity of $f(z)$ and $\lim_{z\to z_0}(z-z_0)f(z) \neq 0$, then z_0 is a pole.

Solution. This is **FALSE**. Let $f(z) = \frac{e^{1/z}}{z}$. Then $z_0 = z$ is an isolated singularity of $f(z)$. In fact, it is an essential singularity since the Laurent series on $0 < |z| < \infty$ is given by

$$\frac{e^{1/z}}{z} = \frac{1}{z}\sum_{n=0}^{\infty} \frac{1}{n!z^n} = \frac{1}{z} + \frac{1}{z^2} + \frac{1}{2z^3} + \frac{1}{6z^4} + \cdots.$$

Yet, $\lim_{z\to 0} zf(z) = \lim_{z\to 0} e^{1/z} \neq 0$. In fact, the limit does not exist. One way to prove this would be consider the sequence $z_n = \frac{1}{n\pi i}$ which satisfies $\lim_{n\to\infty} z_n = 0$. If the limit exists, then by continuity of the exponential we should have

$$\lim_{z\to 0} e^{1/z} = e^{\lim_{z\to 0} 1/z} = e^{\lim_{n\to\infty} 1/z_n} = \lim_{n\to\infty} e^{1/z_n}.$$

But

$$e^{1/z_n} = e^{n\pi i} = \begin{cases} 1, & n \text{ even} \\ -1, & n \text{ odd.} \end{cases}$$

Clearly, $\lim_{n\to\infty} e^{1/z_n}$ does not converge. □

(v) Let $0 < R < S$. If $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ converges to a function $f(z)$ for all $z \in D_R(z_0)$ and $\sum_{n=0}^{\infty} b_n(z-z_0)^n$ converges to $f(z)$ for all $z \in D_S(z_0)$, then $a_n = b_n$ for all $n \in \mathbb{N}$.

Solution. This is **TRUE**. Since $D_R(z_0) \subseteq D_S(z_0)$, it follows that $f(z) = \sum_{n=0}^{\infty} b_n(z-z_0)^n$ for all $z \in D_R(z_0)$. By uniqueness of Taylor series, $a_n = b_n$ for all $n \in \mathbb{N}$. □

(vi) If f is analytic and nonzero at every point on and interior to the unit circle C, then $\int_C \frac{f'(z)}{f(z)}\,dz = 0$.

Solution. This is **TRUE**. Let Z be the number of zeros of $f(z)$ interior to C and let P be the number of poles of f interior to C. Since $f(z) \neq 0$ for all $z \in \operatorname{int} C$, $Z = 0$. Since f is analytic interior to C, $P = 0$. By the argument principle,

$$\begin{aligned} \int_C \frac{f'(z)}{f(z)}\,dz &= \int_{f(C)} \frac{1}{z}\,dz \\ &= Z - P \\ &= 0. \end{aligned}$$

□

(vii) If a function $f(z)$ has an isolated singularity z_0 that is not removable, then $\operatorname{Res}_{z=z_0} f(z) \neq 0$.

Solution. This is **FALSE**. See PSet8 P2(b) for a counterexample. $\square$

F2. (**4 points**) The function $\frac{ze^{\frac{1}{z}}}{z-1}$ is analytic on the annulus $1 < |z| < \infty$ and the Laurent series there is given by

$$\frac{ze^{\frac{1}{z}}}{z-1} = 1 + \frac{2}{z} + \frac{2}{z^2} + \frac{5}{2z^3} + \frac{8}{3z^4} + \cdots.$$

Can you conclude from this that $\operatorname{Res}_{z=0} \frac{ze^{\frac{1}{z}}}{z-1} = 2$? Briefly explain why or why not.

Solution. **NO**. It would be incorrect to make the stated conclusion. Recall the definition of residue: suppose $f(z)$ has an isolated singularity at z_0. By definition, this means that $f(z)$ is analytic on a deleted neighborhood $0 < |z - z_0| < R$ of z_0 and therefore, by Laurent's theorem, has a Laurent series expansion on this annulus:

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} b_n (z - z_0)^{-n}.$$

The residue of $f(z)$ at $z = z_0$ is then defined to be $\operatorname{Res}_{z=z_0} f(z) = b_1$.

But in this problem, although $z_0 = 0$ is an isolated singularity of $\frac{ze^{\frac{1}{z}}}{z-1}$, we are given the Laurent series of $\frac{ze^{\frac{1}{z}}}{z-1}$ on the annulus $1 < |z| < \infty$ which is NOT a deleted neighborhood of $z_0 = 0$. In order to compute the residue, we would need to compute the Laurent series expansion on the domain $0 < |z| < 1$ instead. $\square$

F3. (**6 points**) Let $f(z) = \frac{z}{(z-i)(z-2i)}$ and consider the partial fraction decomposition

$$\frac{z}{(z-i)(z-2i)} = \frac{2}{z-2i} - \frac{1}{z-i}.$$

Compute the Laurent series for $f(z)$ on each annulus:

(a) $0 < |z| < 1$;

(b) $1 < |z| < 2$;

(c) $2 < |z| < \infty$.

Solution. (a) If $0 < |z| < 1$, then we have $\left|\frac{z}{2i}\right| = \frac{|z|}{2} < \frac{1}{2} < 1$. Hence

$$\frac{2}{z-2i} = i \cdot \frac{1}{1 - \frac{z}{2i}} = i \sum_{n=0}^{\infty} \frac{z^n}{2^n i^n} = \sum_{n=0}^{\infty} \frac{z^n}{2^n i^{n-1}}.$$

Similarly, $\left|\frac{z}{i}\right| < 1$. Therefore,

$$\frac{1}{z-i} = i \left(\frac{1}{1 - \frac{z}{i}} \right) = i \sum_{n=0}^{\infty} \frac{z^n}{i^n} = \sum_{n=0}^{\infty} \frac{z^n}{i^{n-1}}.$$

Therefore, on $0 < |z| < 1$,

$$\frac{z}{(z-i)(z-2i)} = \sum_{n=0}^{\infty} \frac{z^n}{2^n i^{n-1}} - \sum_{n=0}^{\infty} \frac{z^n}{i^{n-1}} = \sum_{n=0}^{\infty} (2^{-n} - 1) \frac{z^n}{i^{n-1}}.$$

(b) If $1 < |z| < 2$, then $\left|\frac{z}{2i}\right| = \frac{|z|}{2} < \frac{2}{2} = 1$. Therefore,

$$\frac{2}{z-2i} = \sum_{n=0}^{\infty} \frac{z^n}{2^n i^{n-1}}.$$

as in (a). Moreover, $\left|\frac{i}{z}\right| = \frac{1}{|z|} < 1$ so we get

$$\frac{1}{z-i} = \frac{1}{z} \cdot \left(\frac{1}{1-\frac{i}{z}}\right) = \frac{1}{z}\sum_{n=0}^{\infty} \frac{i^n}{z^n} = \sum_{n=1}^{\infty} \frac{i^{n-1}}{z^n}.$$

Hence, on $1 < |z| < 2$, we have

$$\frac{z}{(z-i)(z-2i)} = \sum_{n=0}^{\infty} \frac{z^n}{2^n i^{n-1}} - \sum_{n=1}^{\infty} \frac{i^{n-1}}{z^n} = \sum_{n=0}^{\infty} \frac{z^n}{2^n i^{n-1}} + \sum_{n=1}^{\infty} \frac{i^{n+1}}{z^n}.$$

(c) If $2 < |z| < \infty$, then we have $\left|\frac{2i}{z}\right| = \left|\frac{2}{z}\right| < 1$ so that

$$\frac{2}{z-2i} = \frac{2}{z}\left(\frac{1}{1-\frac{2i}{z}}\right) = \frac{2}{z}\sum_{n=0}^{\infty} \frac{2^n i^n}{z^n} = \sum_{n=1}^{\infty} \frac{2^n i^{n-1}}{z^n}.$$

Moreover, $\left|\frac{i}{z}\right| = \left|\frac{1}{z}\right| < \left|\frac{2}{z}\right| < 1$ so as in (b),

$$\frac{1}{z-i} = \sum_{n=1}^{\infty} \frac{i^{n-1}}{z^n}.$$

Finally, on $2 < |z| < \infty$ we have

$$\frac{z}{(z-i)(z-2i)} = \sum_{n=1}^{\infty} \frac{2^n i^{n-1}}{z^n} - \sum_{n=1}^{\infty} \frac{i^{n-1}}{z^n} = \sum_{n=1}^{\infty} (2^n - 1)\frac{i^{n-1}}{z^n}.$$

□

F4. (**4 points**) Let $f(z) = \frac{ze^{1/z}}{z^2+5}$. Prove that $f(z)$ has no antiderivative on the domain $D = \{z \in \mathbb{C} : |z| > 5\}$.

Proof. Let C be the circle of radius 6 centered at 0. We will compute the integer $\int_C f(z)\,dz$ using the theorem on residues at ∞. Clearly, $f(z)$ is analytic everywhere on $\mathbb{C}$ except at the singular points $z = \pm\sqrt{5}i, 0$, each of which lie interior to C. Therefore, we have

$$\begin{aligned}\int_C f(z)\,dz &= 2\pi i \operatorname{Res}_{z=0} \frac{1}{z^2} f(1/z)\\ &= 2\pi i \operatorname{Res}_{z=0} \frac{e^z}{z(1+5z^2)}.\end{aligned}$$

To compute the residue, just observe that the function $g(z) = \frac{e^z}{1+5z^2}$ is analytic and therefore has a Taylor series expansion about $z = 0$ whose first term is (by Taylors theorem) $g(0) = 1$. Therefore, the principal part of $\frac{f(z)}{z(1+5z^2)} = \frac{1}{z}g(z)$ at $z = 0$ is simply $\frac{1}{z}$. We conclude that $\operatorname{Res}_{z=0} \frac{f(z)}{z(1+5z^2)} = 1$ and

$$\int_C f(z)\,dz = 2\pi i \neq 0.$$

Another way to compute the residue is to write $p(z) = e^z$ and $q(z) = z + 5z^3$. Then $p(0) = 1$, $q(0) = 0$, and $q'(0) = 1$. By the theorem on simple poles, we can conclude that $\mathrm{Res}_{z=0} \frac{e^z}{z(1+5z^2)} = \frac{p(0)}{q'(0)} = 1$.

Finally, observe that $f(z)$ is continuous on D since $\pm\sqrt{5}i, 0 \notin D$. We conclude by the Fundamental Theorem of Contour Integrals that $f(z)$ has no antiderivative on D. $\square$

F5. (**4 points**) How many zeros (counting orders) does the polynomial

$$p(z) = 2z^5 + 5z^3 + 1$$

have in the annulus $1 \leq |z| < 2$? Justify your computation.

Solution. First, compute the number of zeroes that lie interior to the circle $|z| = 2$ using Rouché. Let $f(z) = 2z^5$ and $g(z) = 5z^3 + 1$ so that $p(z) = f(z) + g(z)$. If $|z| = 2$, then

$$|f(z)| = 2|z|^5 = 2^6$$

and

$$|g(z)| \leq 5|z|^3 + 1 = 5 \cdot 2^3 + 1.$$

Hence,

$$|f(z)| = 2^6 > 5 \cdot 2^3 + 1 \geq |g(z)|$$

for all $|z| = 2$. By Rouché, $f(z) = 2z^5$ and $p(z) = f(z) + g(z)$ have the same number of zeros interior to $|z| = 2$, which is evidently 5.

Next, compute the number of zeros that lie interior to the circle $|z| = 1$. In fact, it was shown in PSet 8 P5 that the number is three. Moreover, the argument used there shows that there are no zeros lying on $|z| = 1$. Therefore, it is appropriate to conclude that the number of zeros of $p(z)$ in the annulus $1 \leq |z| < 2$ is equal to $5 - 3 = 2$. $\square$

F6. (**6 points**) Let $m \geq 1$ be an integer. Follow the steps outlined in Lecture 19 to prove the integration formula[2]

$$\int_{-\infty}^{\infty} \frac{1}{1 + x^{2m}}\, dx = \frac{\pi}{m \sin\left(\frac{\pi}{2m}\right)}.$$

Solution. Let $f(z) = \frac{1}{1+z^{2m}}$. The singularities of $f(z)$ are the roots of the polynomial $1 + z^{2m}$, namely, the $2m$-th roots of -1 which are given by $z_k = e^{i\frac{\pi}{2m}} e^{i\frac{k\pi}{m}}$ for $k = 0, 1, \ldots, 2m-1$. The only singularities with positive imaginary part are given by z_k, $k = 0, 1, \ldots, m-1$. To compute the improper integral, we will integrate $f(z)$ over the semicircular contour (with $R > 1$) described in the lecture. According to the Residue Theorem, we have

$$\int_{-R}^{R} f(x)\, dx = 2\pi i \sum_{k=0}^{m-1} \mathrm{Res}_{z=z_k} f(z) - \int_{C_R} f(z)\, dz.$$

First, we compute the sum of the residues. The singularites of $f(z)$ are distinct roots of a polynomial, so I suspect they are simple poles. In fact, letting $p(z) = 1$, $q(z) = 1 + z^{2m}$, we see that $p(z_k) = 1$, $q(z_k) = 0$, and $q'(z_k) = 2mz_k^{2m-1} \neq 0$. Therefore, by the theorem on simple poles, the singularities are simple with residue given by

$$\mathrm{Res}_{z=z_k} f(z) = \frac{p(z_k)}{q'(z_k)} = \frac{1}{2mz_k^{2m-1}} = -\frac{z_k}{2m}.$$

[2]Hint: to compute the sum of residues, use the formula $\sum_{k=0}^{m-1} w^k = \frac{1-w^m}{1-w}$.

To make the computation easier to read, lets write $\alpha = \frac{\pi}{2m}$ so that $z_k = e^{i\alpha}e^{i2k}$. Then we have

$$\begin{aligned}
2\pi i \sum_{k=0}^{m-1} \operatorname{Res}_{z=z_k} f(z) &= -\frac{\pi i}{m} \sum_{k=0}^{m-1} e^{i\alpha} e^{i2k\alpha} \\
&= -\frac{\pi i}{m} e^{i\alpha} \sum_{k=0}^{m-1} (e^{i2\alpha})^k \\
&= -\frac{\pi i}{m} e^{i\alpha} \cdot \frac{1 - e^{i2m\alpha}}{1 - e^{i2\alpha}} && \text{(see the hint)} \\
&= -\frac{2\pi i}{m} \cdot \frac{e^{i\alpha}}{1 - e^{i2\alpha}} \cdot \frac{-e^{-i\alpha}}{-e^{-i\alpha}} && (e^{i2m\alpha} = e^{i\pi} = -1) \\
&= \frac{2\pi i}{m} \cdot \frac{1}{e^{i\alpha} - e^{-i\alpha}} \\
&= \frac{\pi}{m} \cdot \frac{1}{\sin\alpha} \\
&= \frac{\pi}{m \sin\left(\frac{\pi}{2m}\right)}.
\end{aligned}$$

Next, we prove that $\lim_{R\to\infty} \int_{C_R} f(z)\,dz = 0$. Using the Triangle Inequality for Contour Integrals, we have

$$\begin{aligned}
\left| \int_{C_R} \frac{1}{1+z^{2m}}\,dz \right| &\le \pi R \cdot \max_{|z|=R} \frac{1}{|1+z^{2m}|} \\
&\le \pi R \cdot \max_{|z|=R} \frac{1}{||z|^{2m} - 1|} \\
&= \frac{\pi R}{R^{2m} - 1}.
\end{aligned}$$

The second inequality follows from the reverse triangle inequality: $|1 + z^{2m}| \ge ||z|^{2m} - 1|$ and the last equality from the fact that $|z| = R$ and $R > 1$. Since $\frac{\pi R}{R^{2m}-1} \to 0$ as $R \to \infty$ we have proved the claim. Putting all of this together, we see that

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{1}{1+x^{2m}}\,dx &= \text{P.V.} \int_{-\infty}^{\infty} \frac{1}{1+x^{2m}}\,dx && \text{(since the integrand is even)} \\
&= \lim_{R\to\infty} \int_{-R}^{R} \frac{1}{1+x^{2m}}\,dx \\
&= \lim_{R\to\infty} 2\pi i \sum_{k=0}^{m-1} \operatorname{Res}_{z=z_k} f(z) - \lim_{R\to\infty} \int_{C_R} f(z)\,dz \\
&= \frac{\pi}{m \sin\left(\frac{\pi}{2m}\right)}.
\end{aligned}$$

This completes the solution. □

F7. (Optional.)[3] Compute $\operatorname{Res}_{z=0} \frac{e^{\frac{1}{z}}}{1-z}$. Justify your computations.

Solution. We offer two possible solutions.

[3] Worth 4 points of extra credit, added to your score on this exam.

The "naive" approach: In this solution, we are implicitly assuming that the formal multiplication of a Taylor series determines the same coefficients as those guaranteed by Laurent's theorem. The justification for this is beyond the scope of an introductory course - but the solution is still interesting.

The functions $\frac{1}{1-z}$ and $e^{1/z}$ are both analytic on $0 < |z| < 1$. We have

$$\begin{aligned} \frac{e^{\frac{1}{z}}}{1-z} &= \frac{1}{1-z} \cdot e^{1/z} \\ &= \left(1 + z + z^2 + z^3 + \cdots\right) \cdot \left(\frac{1}{0!} + \frac{1}{1!z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \cdots\right). \end{aligned}$$

It is easy to see that, after performing the formal multiplication, the coefficient of $\frac{1}{z}$ is given by

$$\mathrm{Res}_{z=0} \frac{e^{\frac{1}{z}}}{1-z} = \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n!} = \sum_{n=0}^{\infty} \frac{1}{n!} - 1 = e - 1.$$

The formal multiplication can be justified by realizing that $\frac{1}{0!} + \frac{1}{1!z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \cdots$ is a Taylor series in $w = \frac{1}{z}$.

A better approach: This solution uses the powerful theorem for integrating power series, which we did prove in the lecture. Let C be the positively oriented circle with radius $1/2$. Since $f(z) = \frac{e^{\frac{1}{z}}}{1-z}$ is analytic on the annulus $0 < |z| < 1$, according to Laurent's theorem the desired residue is given by $\mathrm{Res}_{z=0} \frac{e^{\frac{1}{z}}}{1-z} = \frac{1}{2\pi i} \int_C \frac{e^{\frac{1}{z}}}{1-z}\, dz$. Since C is contained in the disk of convergence of the power series $\sum_{n=0}^{\infty} z^n$ and $e^{1/z}$ is continuous on C, we have

$$\begin{aligned} \mathrm{Res}_{z=0} \frac{e^{\frac{1}{z}}}{1-z} &= \frac{1}{2\pi i} \int_C \frac{e^{\frac{1}{z}}}{1-z}\, dz \\ &= \frac{1}{2\pi i} \int_C e^{\frac{1}{z}} \sum_{n=0}^{\infty} z^n\, dz \\ &= \frac{1}{2\pi i} \sum_{n=0}^{\infty} \int_C z^n e^{\frac{1}{z}}\, dz && \text{(Integrating Power Series)} \\ &= \sum_{n=0}^{\infty} \frac{1}{(n+1)!} && \text{(PSet 7 P5(b))} \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} - 1 \\ &= e - 1. \end{aligned}$$

This ends the solution. □